Magnetohydrodynamic free convection

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(Received 23 September 1963)

The flow of an electrically conducting fluid up a hot vertical plate in the presence of a strong magnetic field normal to the plate is considered. A solution is developed based on the idea of matching 'outer' and 'inner' solutions in the moving layer of fluid. An approximate Pohlhausen method of solution is also given which yields results in fairly good agreement with the exact analysis.

1. Introduction

Several recent authors, Geshuni & Zhukovitski (1958), Lykoudis (1962), Gupta (1960, 1962), Sparrow & Cess (1961) and Singh & Cowling (1963*a*, *b*), have considered the problem of free convection of an electrically conducting fluid in the presence of a magnetic field. Of these papers the latter two contain perhaps the most penetrating analysis of the general flow properties in such a physical situation. In the first of their papers Singh & Cowling discuss the two-dimensional flow of a fluid of small kinetic viscosity ν up a hot vertical plate normal to which a uniform magnetic field is applied. In the second the flow in a closed rectangular box, whose vertical walls are normal to the field and maintained at different temperatures, is examined. The present work is an extension of the first of these papers.

In their paper Singh & Cowling show that, regardless of the strength of the applied magnetic field, there will always be a region in the neighbourhood of the leading edge of the plate where forces of electromagnetic origin are not important, whilst at large distances from the leading edge these magnetic forces dominate. In each of these regions they give a similarity solution of the equations of motion. For the flow sufficiently near the leading edge the solution is the well known non-magnetic free-convection solution; Sparrow & Cess (1961) take this solution as the first term in a series expansion about the leading edge. Singh & Cowling also develop an approximate Pohlhausen method of solution based on an integrated form of the equations of motion and from this recover the main features of their similarity solutions. Although in their work they are mainly concerned with the case where the wall temperature and temperature of the ambient fluid are constant, they also discuss the principle features of the flow induced by other temperature conditions.

In the present paper we consider the flow up a vertical plate maintained at a constant temperature greater than the constant temperature of the surrounding fluid in the presence of a strong magnetic field normal to the plate. Of the two parameters associated with the electromagnetic features of the flow we assume that the Hartmann number M ($=(\sigma/\rho_0\nu)^{\frac{1}{2}}H_0L$ where σ is the conductivity, ρ_0 the density and ν the kinematic viscosity of the fluid, H_0 the magnitude of the applied magnetic field and L a typical length, say the length of the plate) is much greater than unity, and that the magnetic Reynolds number R_M ($= 4\pi\kappa\sigma$ where κ is the thermal diffusivity) is much less than unity. Consequently (i) the electromagnetic forces are important everywhere except in the immediate neighbourhood of the leading edge of the plate, and (ii) perturbations to the basic normal field may be ignored.

The starting-point of the analysis is the similarity solution given by Singh & Cowling. This solution, which is based on the assumption that the viscous and inertia forces may be neglected in comparison with the buoyancy and magnetic forces, ignores the presence of a very thin layer of fluid adjacent to the wall in which viscous forces are important, and consequently the no-slip condition at the wall is violated. Thus, although the dominant terms in the volume flow of fluid up the plate and the heat transfer across the wall are given by this solution no estimate of the skin friction can be made. The introduction of an inner boundary layer which enables the no-slip condition at the wall to be satisfied and matches with the similarity solution remedies this deficiency. The thickness of this inner boundary layer is shown to be $O(M^{-1})$ and in this respect is similar to the boundary layers to be found, when M is large, on the walls in Hartmann flow, and on the walls normal to the applied field in the flow down tubes of rectangular cross-section considered by Shercliff (1953). The similarity solution together with the above solution in the inner boundary layer forms the basis for extending the solution by introducing a series of 'outer' and 'inner' solutions between which a matching procedure is effected. At the third stage in the outer solution a difficulty arises necessitating the introduction of logarithmic terms. Also at this stage an indeterminacy arises manifesting itself as a shift of origin; this may be attributed to the fact that our solution does not hold at the leading edge of the plate and so does not satisfy the boundary conditions imposed there and in this sense the solution may only be regarded as asymptotic. The solution is not continued beyond this stage. The first three terms of the series for the volume flow of fluid up the plate and the skin friction together with the first two terms for the heat transfer across the plate are deduced from the solution.

In the approximate solution which they give using a Pohlhausen method Singh & Cowling choose temperature and velocity profiles which approximate closely to their similarity solution. Thus, again, the inner boundary layer is ignored and the viscous term dropped from the integrated form of the equations of motion. The approximate solution then predicts only those features of the flow given by the similarity solution. In the present work a Pohlhausen solution is given based on a velocity profile which takes account of the inner boundary layer of thickness $O(M^{-1})$. All the terms in the integrated form of the momentum equation are retained and the main features of the solution described above are recovered. The accuracy of the results compares favourably with the exact analysis.

It is assumed throughout that the fluid is non-magnetic so that its permeability is unity.

2. Equations of motion

For the derivation of the equations of motion for the problem under consideration the reader is referred to the paper by Singh & Cowling (1963*a*). In dimensionless form the momentum and energy equations take the forms

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\lambda\frac{\partial^3\psi}{\partial y^3}=A_\kappa\theta-M_\kappa^2\frac{\partial\psi}{\partial y}, \tag{1}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}=\frac{\partial^2\theta}{\partial y^2}, \tag{2}$$

with boundary conditions

$$\left.\begin{array}{ll} \psi=\partial\psi/\partial y=0, \quad \theta=1 \quad \text{at} \quad y=0 \\ \text{and} \quad \partial\psi/\partial y=\theta=0 \quad \text{at} \quad y=\infty. \end{array}\right\} \tag{3}$$

The boundary conditions on the dimensionless temperature θ, defined as $\theta=(T-T_0)/(T_1-T_0)$ where T is the temperature, are based on the assumption that the wall and ambient fluid are maintained at constant temperatures T_1 and T_0 respectively ($T_1>T_0$). To complete the specification of the problem we require that

$$\theta=\partial\psi/\partial y=0 \quad \text{at} \quad x=0, \quad y>0, \tag{4}$$

these conditions being necessary since the governing equations are parabolic. A representative length L and velocity κ/L together with the temperature (T_1-T_0) are used in rendering these equations dimensionless The dimensionless velocities along and normal to the wall are related to the stream function ψ by

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x},$$

respectively, where x and y are measured along and perpendicular to the plate. (In this respect the notation differs from that used by Singh & Cowling who measure y along the plate.) The Prandtl number $\lambda=\nu/\kappa$; the parameters A_κ and M_κ are respectively the modified Grashof and Hartmann numbers. Thus the ordinary Grashof number ($=\alpha_0 g(T_1-T_0)L^3/\nu^2$ where α_0 is the coefficient of expansion and g the acceleration due to gravity) and Hartmann number are $\lambda^{-2}A_\kappa$ and $\lambda^{-\frac{1}{2}}M_\kappa$. It may be noted that in formulating the equations of motion the Boussinesq approximation, where variations of density are taken account of only in the buoyancy term of the momentum equation, has been employed. We also define $B=A_\kappa M_\kappa^{-2}$. Perturbations to the basic magnetic field which is normal to the plate are $O(R_M)$ and are ignored. The electric field $\mathbf{E}$ is assumed to be zero.

In the present work the quantities λ and A_κ are assumed to be $O(1)$ but M (and consequently M_κ) are assumed large, consistent with an applied field of very great strength; consequently $B \ll 1$. Under these conditions the magnetic drag force tending to oppose motion across the lines of force is of the same order of magnitude as the buoyancy force, and dominates the inertia and viscous forces everywhere except (i) in a very narrow viscous sublayer where viscous forces must be important, and (ii) in the immediate neighbourhood of the leading edge of the plate. Singh & Cowling show that in the latter region the flow approximates to that in the absence of a magnetic field. Sparrow & Cess (1961) have obtained a series solution, valid in this region, by expanding about the basic non-magnetic solution; it will not be discussed further here.

For the region in which magnetic drag dominates Singh & Cowling obtain a similarity solution as follows. Write

$$\left.\begin{aligned} \psi &= B^{\frac{1}{2}}x^{\frac{1}{2}}F_0(\eta), \\ \theta &= \theta_0(\eta), \quad \text{where} \quad \eta = B^{\frac{1}{2}}y/x^{\frac{1}{2}}, \end{aligned}\right\} \tag{5}$$

and then it follows from (1) that $\theta_0 = F_0'$, and from (2) that

$$F_0''' + \tfrac{1}{2}F_0F_0'' = 0, \tag{6}$$

where the primes denote differentiation with respect to η. Since the order of the governing equations has been reduced by two not all of the boundary conditions (3) can be satisfied so the no-slip condition and the condition at infinity on ψ are abandoned. The other three conditions give

$$F_0(0) = F_0'(\infty) = 0, \quad F_0'(0) = 1. \tag{7}$$

Singh & Cowling have integrated (6) numerically under the boundary conditions (7) (although it may be noted that the solution is readily derived from tables published in connexion with the problem of the mixing of a non-conducting fluid at rest with a moving stream of the same fluid, see e.g. Christian 1961). Their principle results may be summarized as

$$\left.\begin{aligned} &F_0(\infty) = 1{\cdot}616, \\ &F_0''(0) = \gamma = -0{\cdot}4437, \\ \text{and for small } \eta, \quad &F_0(\eta) = \eta + \frac{\gamma\eta^2}{2!} - \frac{\gamma}{2}\frac{\eta^4}{4!} - \frac{\gamma^2}{2}\frac{\eta^5}{5!} + O(\eta^6). \end{aligned}\right\} \tag{8}$$

Although this solution gives the dominant term in both the volume flow up the plate and the heat transfer across the plate, it may be criticized on the grounds that the no-slip condition has been violated. The purpose of the present paper is to show how an inner boundary layer may be introduced to reduce the fluid velocity at the wall to rest enabling the solution, the leading term of which is given above, to be developed further.

3. Solution of equations

It is more convenient here to work in terms of the variable

$$\xi = x/B, \tag{9}$$

in terms of which equations (1) and (2) become

$$\frac{1}{A_\kappa B}\left\{\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial\xi\partial y} - \frac{\partial\psi}{\partial\xi}\frac{\partial^2\psi}{\partial y^2}\right\} - \frac{\lambda}{A_\kappa}\frac{\partial^3\psi}{\partial y^3} = \theta - \frac{1}{B}\frac{\partial\psi}{\partial y}, \tag{10}$$

$$\frac{1}{B}\left\{\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\theta}{\partial y}\right\} = \frac{\partial^2\theta}{\partial y^2}. \tag{11}$$

The method of solution adopted is that of matching solutions which are valid in an inner viscous layer to appropriate outer solutions of which the first has been given by Singh & Cowling. To establish the form of the equations appropriate to the inner layer we note that in this layer the viscous forces are of the same order of magnitude as the buoyancy and magnetic forces, and that $\theta = O(1)$, $\partial\psi/\partial y = O(B)$ in order that the inner solution will match with the solution given by Singh & Cowling. These considerations lead to the following transformation (where now 'inner' variables are denoted by a bar)

$$\left.\begin{aligned} \psi &= B^{\frac{3}{2}}\,\bar{\psi}, & y &= B^{\frac{1}{2}}\bar{y}, \\ \theta &= \bar{\theta}, & \xi &= \bar{\xi}. \end{aligned}\right\} \tag{12}$$

Hence, from (10), (11) and (12), the form of the equations which we shall use in the inner layer are

$$\frac{\lambda}{A_\kappa}\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3} - \frac{\partial\bar{\psi}}{\partial\bar{y}} + \bar{\theta} = \frac{B}{A_\kappa}\left\{\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{\xi}\partial\bar{y}} - \frac{\partial\bar{\psi}}{\partial\bar{\xi}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\right\}, \tag{13}$$

and

$$\frac{\partial^2\bar{\theta}}{\partial\bar{y}^2} = B\left\{\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial\bar{\theta}}{\partial\bar{\xi}} - \frac{\partial\bar{\psi}}{\partial\bar{\xi}}\frac{\partial\bar{\theta}}{\partial\bar{y}}\right\}, \tag{14}$$

with boundary conditions,

$$\bar{\psi} = \frac{\partial\bar{\psi}}{\partial\bar{y}} = 0, \quad \bar{\theta} = 1 \quad \text{at} \quad \bar{y} = 0, \tag{15}$$

and as $\bar{y} \to \infty$ the solution must match with the outer solution as $\eta \to 0$. It is noted that the leading term of the outer solution is the solution given by Singh & Cowling which is summarized in § 2.

A solution to the equations for the inner layer is sought in the form

$$\left.\begin{aligned} \bar{\psi} &= \bar{\psi}_0 + B\bar{\psi}_1 + O(B^2), \\ \bar{\theta} &= \bar{\theta}_0 + B\bar{\theta}_1 + O(B^2), \end{aligned}\right\} \tag{16}$$

and we see that $\bar{\psi}_0$ and $\bar{\theta}_0$ satisfy

$$\left.\begin{aligned} \frac{\partial^2\bar{\theta}_0}{\partial\bar{y}^2} &= 0, \\ \frac{\lambda}{A_\kappa}\frac{\partial^3\bar{\psi}_0}{\partial\bar{y}^3} - \frac{\partial\bar{\psi}_0}{\partial\bar{y}} + \bar{\theta}_0 &= 0. \end{aligned}\right\} \tag{17}$$

The solutions of equations (17) which satisfy the boundary conditions at the wall are

$$\left.\begin{aligned} \bar{\theta}_0 &= 1 + a(\bar{\xi})\,\bar{y}, \\ \bar{\psi}_0 &= \bar{y} + \tfrac{1}{2}a(\bar{\xi})\,\bar{y}^2 - \left(\frac{\lambda}{A_\kappa}\right)^{\frac{1}{2}}\left\{1 - \exp\left[-\left(\frac{A_\kappa}{\lambda}\right)^{\frac{1}{2}}\bar{y}\right]\right\}, \end{aligned}\right\} \tag{18}$$

where $a(\bar{\xi})$ is an arbitrary function of $\bar{\xi}$ and is to be determined from the matching condition as $\bar{y} \to \infty$. From (12) and the relationship between the various parameters we see that the last term in (18) can be written as $\exp(-My)$ showing that the thickness of the inner boundary layer is $O(M^{-1})$. It may be noted that for $M \gg 1$ boundary layers of thickness $O(M^{-1})$ are formed on the walls normal to the applied field in both Hartmann flow (see Cowling 1957), and the flow down a pipe of rectangular cross-section studied by Shercliff (1953).

The arbitrary function $a(\bar{\xi})$ in (18) may be determined by letting $\bar{y} \to \infty$ in $\bar{\psi}_0$ and comparing with ψ, in (5), as $\eta \to 0$. The first term matches automatically and for the second term to match we require

$$a(\bar{\xi}) = \gamma B^{\frac{1}{2}}\bar{\xi}^{-\frac{1}{2}}. \tag{19}$$

No term in the outer solution as given by (5) matches with the constant term in $\bar{\psi}_0$ and consequently we must introduce terms $(\lambda/A_\kappa)^{\frac{1}{2}} B^{\frac{3}{2}} F_1(\eta)$, $(\lambda/A_\kappa)^{\frac{1}{2}} B^{\frac{1}{2}} \xi^{-\frac{1}{2}} \theta_1(\eta)$ into the outer solutions for ψ and θ. It is readily shown that $\theta_1 = F_1'$ and that F_1' satisfies a second-order homogeneous equation with solution $F_1 = -1$. This means that we merely add the constant $-(\lambda/A_\kappa)^{\frac{1}{2}} B^{\frac{3}{2}}$ to the outer solution for ψ at this stage. This constant may be interpreted as the defect of volume flux up the plate due to the presence of the inner boundary layer.

Turning again to the inner solution we see from (13), (14), (16) and (18) that $\bar{\theta}_1$ and $\bar{\psi}_1$ satisfy

$$\left.\begin{aligned} \frac{\partial^2 \bar{\theta}_1}{\partial \bar{y}^2} &= \dot{a}\bar{y} + \tfrac{1}{2} a\dot{a}\bar{y}^2 - \dot{a}\bar{y}e^{-\alpha\bar{y}}, \\ \frac{\lambda}{A_\kappa}\frac{\partial^3 \bar{\psi}_1}{\partial \bar{y}^3} - \frac{\partial \bar{\psi}_1}{\partial \bar{y}} + \bar{\theta}_1 &= \frac{1}{A_\kappa}\left\{\dot{a}\bar{y} + \tfrac{1}{2} a\dot{a}\bar{y}^2 - \dot{a}\bar{y}e^{-\alpha\bar{y}} - \frac{\alpha}{2}\dot{a}\bar{y}^2 e^{-\alpha\bar{y}}\right\}, \end{aligned}\right\} \tag{20}$$

where $\alpha^2 = A_\kappa/\lambda$ and the dots denote differentiation with respect to $\bar{\xi}$. Solutions of the above two equations satisfying

$$\bar{\theta}_1 = \bar{\psi}_1 = \frac{\partial \bar{\psi}_1}{\partial \bar{y}} = 0 \quad \text{at} \quad \bar{y} = 0,$$

are, respectively,

$$\bar{\theta}_1 = \tfrac{1}{6}\dot{a}\bar{y}^3 + \tfrac{1}{24} a\dot{a}\bar{y}^4 + b\bar{y} - \frac{1}{\alpha^2}\dot{a}\bar{y}e^{-\alpha\bar{y}} + \frac{2}{\alpha^3}\dot{a}(1 - e^{-\alpha\bar{y}}), \tag{21}$$

$$\begin{aligned} \bar{\psi}_1 = {} & \frac{\dot{a}}{24}\bar{y}^4 + \frac{a\dot{a}}{120}\bar{y}^5 + \frac{\dot{a}}{2}\beta\bar{y}^2 + \frac{b}{2}\bar{y}^2 + \frac{\beta}{6}a\dot{a}\bar{y}^3 + \frac{2}{\alpha^3}\dot{a}\bar{y} + \frac{\beta}{\alpha^2}a\dot{a}\bar{y} + \frac{\dot{a}}{\alpha^3}\bar{y}e^{-\alpha\bar{y}} \\ & + \beta\dot{a}\left(\frac{\bar{y}^2}{4} + \frac{3\bar{y}}{4\alpha}\right)e^{-\alpha\bar{y}} - \frac{\alpha\dot{a}}{2A_\kappa}\left(\frac{\bar{y}^3}{6} + \frac{3\bar{y}^2}{4\alpha} + \frac{7\bar{y}}{4\alpha^2}\right)e^{-\alpha\bar{y}} + c(e^{-\alpha\bar{y}} - 1), \end{aligned} \tag{22}$$

where $$\beta = A_\kappa^{-1}(\lambda - 1), \quad c(\bar{\xi}) = \frac{\beta}{\alpha^3}a\dot{a} + \frac{3\beta}{4\alpha^2}\dot{a} + \frac{3}{\alpha^4}\dot{a} - \frac{7\dot{a}}{8\alpha^2 A_\kappa},$$

and the arbitrary function $b(\bar{\xi})$ is determined by the matching condition. That the first two terms of (21) and (22) match with the third and fourth terms of the functions $\theta_0 (= F_0')$ and F_0 can be seen by comparing (21) and (22), as $\bar{y} \to \infty$, with the expansion of F_0 for small η given in equation (8). The third term in (21) and the next three terms in (22) will match with terms arising from the next of the outer solutions for θ and ψ, not yet considered. The remaining terms in (21) and (22) which are not exponentially small at infinity need not be considered at this stage.

We must now find the next term in the outer solution. If we substitute the expansions

$$\left.\begin{aligned} \psi &= B\xi^{\frac{3}{4}} F_0(\eta) - (\lambda/A_\kappa)^{\frac{1}{2}} B^{\frac{3}{2}} + \beta B^2 \xi^{-\frac{1}{4}} F_2(\eta) + \ldots, \\ \theta &= \theta_0(\eta) + \beta B \xi^{-1} \theta_2(\eta) + \ldots, \end{aligned}\right\} \tag{23}$$

into equations (10) and (11), we see that the equations satisfied by θ_2 and F_2 may be written

$$\theta_2 = F_2' - F_0''', \tag{24}$$

$$F_2'''' + \tfrac{1}{2} F_0 F_2'' + F_0' F_2' - \tfrac{1}{2} F_0'' F_2 = -\tfrac{1}{2} F_0''^2. \tag{25}$$

The boundary conditions for F_2 are

$$F_2'(\infty) = 0, \tag{26}$$

and $$F_2(0) = F_2'(0) = 0, \tag{27}$$

the latter two determined from the matching condition as $\eta \to 0$. The behaviour, as $\eta \to \infty$, of the homogeneous equation associated with (25) is easily determined as

$$F_2 \sim k_1\eta + k_2 + k_3 e^{-\epsilon\eta}, \tag{28}$$

where $2\epsilon = F_0(\infty)$ and k_1, k_2 and k_3 are constants. Two independent solutions ϕ_1 and ϕ_2 of the homogeneous part of (25), both of which are bounded as $\eta \to \infty$, are

$$\left.\begin{aligned} \phi_1 &= F_0', \\ \phi_2 &= F_0 - \eta F_0'. \end{aligned}\right\} \tag{29}$$

Thus a third solution ϕ_3 which is independent of ϕ_1 and ϕ_2 will be algebraically large at infinity, and without loss of generality we may choose ϕ_3 such that $\phi_3(0) = 1$, $\phi_3'(0) = \phi_3''(0) = 0$. Using the method of variation of parameters a formal solution of (25) satisfying (27) may be obtained which shows that if F_2 is not to be algebraically large at infinity then the integral condition

$$\int_0^\infty (\phi_2\phi_1' - \phi_1\phi_2')\, F_0''^2 \exp\left[\frac{1}{2}\int_0^\eta F_0 d\eta\right] d\eta = 0, \tag{30}$$

must hold. From (29) and the fact that from equation (6) we may write

$$F_0'' = \gamma \exp\left[-\frac{1}{2}\int_0^\eta F_0 d\eta\right], \tag{31}$$

the integral condition (30) becomes

$$\int_0^\infty F_0'' F_0''' d\eta = 0,$$

which is clearly not satisfied since the integral has the value $-\frac{1}{2}\gamma^2$. This difficulty arises because the expansions (23) are incomplete and may be overcome by adding logarithmic terms. Thus to ψ and θ in (23) we add terms $C_2\beta B^2\xi^{-\frac{1}{2}} \log \xi F_{20}(\eta)$ and $C_2\beta B\xi^{-1} \log \xi\, \theta_{20}(\eta)$, respectively, where C_2 is a constant which is to be determined. The equations satisfied by θ_{20} and F_{20} are

$$\theta_{20} = F_{20}', \tag{32}$$

$$F_{20}''' + \tfrac{1}{2}F_0 F_{20}'' + F_0' F_{20}' - \tfrac{1}{2}F_0'' F_{20} = 0, \tag{33}$$

and the boundary conditions for F_{20} are the same as for F_2 given in (26) and (27). Consequently, since C_2 is arbitrary as yet, we may take

$$F_{20} = F_0 - \eta F_0'. \tag{34}$$

The equation (24) for θ_2 is unaltered but the equation for F_2 is now

$$F_2''' + \tfrac{1}{2}F_0 F_2'' + F_0' F_2' - \tfrac{1}{2}F_0'' F_2 = -\tfrac{1}{2}F_0''^2 - C_2 F_0'' F_{20}. \tag{35}$$

Again the formal solution shows that if F_2 is not to be algebraically large at infinity an integral condition analogous to (30) must be satisfied. From (31) and (34) this reduces to

$$C_2 \int_0^\infty \eta F_0''^2 d\eta = \frac{\gamma^2}{4}, \tag{36}$$

and a numerical integration gave $C_2 = 0{\cdot}171$. Although the logarithmic term in the expansion is given uniquely it will be observed that F_2 cannot be so determined since we can always add to F_2 a constant multiple of ϕ_2 given in (29). This corresponds to a shift of origin, the arbitrariness arising because our solution does not satisfy the boundary conditions at $\xi = 0$. In this sense the solution is asymptotic. Similar difficulties, described by Stewartson (1957), arise in other boundary-layer problems. The solution is not considered beyond this stage.

To complete the solution so far it remains to determine the arbitrary function $b(\bar{\xi})$ occurring in (21) and (23). On account of the arbitrariness in F_2 discussed above $b(\bar{\xi})$ cannot be determined uniquely. Matching the outer solution for ψ as $\eta \to 0$ with (22) as $\bar{y} \to \infty$ gives

$$b(\bar{\xi}) = 0{\cdot}076\beta B^{\frac{1}{2}}\bar{\xi}^{-\frac{3}{4}}\log\bar{\xi} + O(B^{\frac{1}{2}}\bar{\xi}^{-\frac{3}{4}}). \tag{37}$$

It may be noted, as indicated earlier, that for sufficiently large M the form of solution discussed here will hold everywhere except in the immediate neighbourhood of the leading edge of the plate.

The principle features of the flow, the mass flow up the plate, the skin friction, and heat transfer across the plate may be deduced from

$$(\psi)_{y=\infty} = 1{\cdot}616B\xi^{\frac{1}{2}} - \left(\frac{\lambda}{A_\kappa}\right)^{\frac{1}{2}} B^{\frac{3}{2}} + 0{\cdot}276\beta B^2\xi^{-\frac{1}{2}}\log\xi + O(B^2\xi^{-\frac{1}{2}}), \tag{38}$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{A_\kappa}{\lambda}\right)^{\frac{1}{2}} B^{\frac{1}{2}} - 0{\cdot}444B\xi^{-\frac{1}{2}} + 0{\cdot}076\beta B^2\xi^{-\frac{3}{2}}\log\xi + O(B^2\xi^{-\frac{3}{2}}), \tag{39}$$

$$\left(\frac{\partial \theta}{\partial y}\right)_{y=0} = -0{\cdot}444\xi^{-\frac{1}{2}} + 0{\cdot}076\,\beta B\xi^{-\frac{3}{2}}\log\xi + O(B\xi^{-\frac{3}{2}}). \tag{40}$$

In conclusion we note that if $\lambda = 1$, $\beta = 0$ and logarithmic terms do not occur at this stage.

4. Approximate method of solution

Singh & Cowling in their paper give an approximate Pohlhausen solution based on an integrated form of the equations of motion. These are obtained by integrating (1) and (2) with respect to y from $y = 0$ to $y = \delta$ to give, when wall and ambient fluid temperatures are constant,

$$\frac{d}{dx}\int_0^\delta u^2 dy = A_\kappa \int_0^\delta \theta\, dy - M_\kappa^2 \int_0^\delta u\, dy - \lambda\left(\frac{\partial u}{dy}\right)_{y=0}, \tag{41}$$

and

$$\frac{d}{dx}\int_0^\delta u\theta\, dy = -\left(\frac{\partial \theta}{\partial y}\right)_{y=0}. \tag{42}$$

The edge of the layer of fluid in motion is taken to be $y = \delta$. The polynomial expressions assumed for u and θ by Singh & Cowling when the magnetic drag is dominant are

$$u = V(x)\,(1 - y/\delta)^n, \tag{43}$$

$$\theta = (1 - y/\delta)^n, \tag{44}$$

where $V(x)$, $\delta(x)$ are to be determined from (41) and (42). However, the expression (43) for u does not satisfy the no-slip condition at the wall and indeed, in their analysis, Singh & Cowling omit the viscous term in (41) altogether. In this way, as with their similarity solution, only the first terms in the expressions for $(\psi)_{y=\infty}$ and $(\partial\theta/\partial y)_{y=0}$ can be estimated.

This solution may be improved and the viscous term in (41) included by choosing a more appropriate form for u, embodying the inner layer which has thickness $O(M^{-1})$. Thus we write

$$u = V(x)(1-y/\delta)^n - V(x)\,e^{-My}, \tag{45}$$

and substituting for θ and u in equations (41) and (42) from (44) and (45) we have the following two equations to solve for $V(x)$ and $\delta(x)$:

$$\frac{(n+1)}{(2n+1)}\frac{d}{dx}[V^2\delta\{1+O[(M\delta)^{-1}]\}] = A_\kappa\delta - M_\kappa^2 V\delta + n(n+1)\lambda\frac{V}{\delta}, \tag{46}$$

$$\frac{d}{dx}[V\delta\{1+O[(M\delta)^{-1}]\}] = \frac{n(2n+1)}{\delta}. \tag{47}$$

Following Singh & Cowling we now introduce new variables x', V' and δ' where

$$x = \frac{(n+1)A_\kappa}{(2n+1)M_\kappa^4}x', \quad \delta = \frac{[n(n+1)]^{\frac{1}{2}}}{M_\kappa}\delta', \quad V = \frac{A_\kappa}{M_\kappa^2}V', \tag{48}$$

and writing
$$V'\delta' = z, \quad V'^2\delta' = w, \tag{49}$$

we see from (46), (47) and (49) that w and z are related by the first-order equation

$$\frac{dw}{dz} = \frac{z^3(z-w)}{w^2} + \lambda\frac{w}{z} + R_1(z), \tag{50}$$

where for large z, the case under consideration, it can be verified *a posteriori* that $R_1 = O(z^{-3})$. The equation studied by Singh & Cowling is obtained from (50) by setting λ and R_1 equal to zero. The relationship between x' and z is, using equations (47) and (48),

$$\frac{dx'}{dz} = \frac{z^2}{w} + R_2(z), \tag{51}$$

where for large z, $R_2 = O(z^{-2})$. For large z we have, from equation (50),

$$w \sim z + (\lambda-1)/z, \tag{52}$$

and so, from (51),
$$\frac{dx'}{dz} \sim z - (\lambda-1)/z, \tag{53}$$

giving
$$x' \sim \tfrac{1}{2}z^2 - (\lambda-1)\log z + C, \tag{54}$$

where the arbitrary constant C depends on initial conditions and is analogous to the indeterminacy discussed in the previous section. Inverting (54) we get

$$z = (2x')^{\frac{1}{2}} + 2^{-\frac{3}{2}}(\lambda-1)\,x'^{-\frac{1}{2}}\log x' + O(x'^{-\frac{1}{2}}), \tag{55}$$

and the constant C now only occurs in the $O(x'^{-\frac{1}{2}})$ term. The quantities V' and δ' are now given, from equations (49), (52) and (55) as

$$V' = 1 + O(x'^{-1}), \tag{56}$$

$$\delta' = (2x')^{\frac{1}{2}} + 2^{-\frac{3}{2}}(\lambda-1)\,x'^{-\frac{1}{2}}\log x' + O(x'^{-\frac{1}{2}}). \tag{57}$$

In terms of the variable ξ defined in (9), the expressions necessary for calculating the mass flow up the plate, the skin friction, and the heat transfer across the plate are, using equations (44), (45), (48), (56) and (57),

$$(\psi)_{y=\infty} = \frac{[2n(2n+1)]^{\frac{1}{2}}}{(n+1)} B\xi^{\frac{1}{2}} - \left(\frac{\lambda}{A_\kappa}\right)^{\frac{1}{2}} B^{\frac{3}{2}} + \frac{\beta}{2}\left[\frac{n}{2(2n+1)}\right]^{\frac{1}{2}} B^2\xi^{-\frac{1}{2}}\log\xi + O(B^2\xi^{-\frac{1}{2}}), \tag{58}$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{A_\kappa}{\lambda}\right)^{\frac{1}{2}} B^{\frac{1}{2}} - \left[\frac{n}{2(2n+1)}\right]^{\frac{1}{2}} B\xi^{-\frac{1}{2}} + \frac{\beta}{4}\left[\frac{n(n+1)^2}{2(2n+1)^3}\right]^{\frac{1}{2}} B^2\xi^{-\frac{3}{2}}\log\xi + O(B^2\xi^{-\frac{3}{2}}), \tag{59}$$

$$\left(\frac{\partial \theta}{\partial y}\right)_{y=0} = -\left[\frac{n}{2(2n+1)}\right]^{\frac{1}{2}} \xi^{-\frac{1}{2}} + \frac{\beta}{4}\left[\frac{n(n+1)^2}{2(2n+1)^3}\right]^{\frac{1}{2}} B\xi^{-\frac{3}{2}}\log\xi + O(B\xi^{-\frac{3}{2}}). \tag{60}$$

Singh & Cowling only considered the polynomial forms given by $n = 2, 3$ and we set out below the expressions (58) and (59) in these two cases. The numerical coefficients in equation (60) are easily deduced from (59):

(i) $n = 2$

$$\left.\begin{aligned}(\psi)_{y=\infty} &= 1{\cdot}491B\xi^{\frac{1}{2}} - \left(\frac{\lambda}{A_\kappa}\right)^{\frac{1}{2}} B^{\frac{3}{2}} + 0{\cdot}224\beta B^2\xi^{-\frac{1}{2}}\log\xi + O(B^2\xi^{-\frac{1}{2}}),\\ \left(\frac{\partial u}{\partial y}\right)_{y=0} &= \left(\frac{A_\kappa}{\lambda}\right)^{\frac{1}{2}} B^{\frac{1}{2}} - 0{\cdot}447B\xi^{-\frac{1}{2}} + 0{\cdot}067\beta B^2\xi^{-\frac{3}{2}}\log\xi + O(B^2\xi^{-\frac{3}{2}}),\end{aligned}\right\} \tag{61}$$

(ii) $n = 3$

$$\left.\begin{aligned}(\psi)_{y=\infty} &= 1{\cdot}620B\xi^{\frac{1}{2}} - \left(\frac{\lambda}{A_\kappa}\right)^{\frac{1}{2}} B^{\frac{3}{2}} + 0{\cdot}232\beta B^2\xi^{-\frac{1}{2}}\log\xi + O(B^2\xi^{-\frac{1}{2}}),\\ \left(\frac{\partial u}{\partial y}\right)_{y=0} &= \left(\frac{A_\kappa}{\lambda}\right)^{\frac{1}{2}} B^{\frac{1}{2}} - 0{\cdot}463B\xi^{-\frac{1}{2}} + 0{\cdot}066\beta B^2\xi^{-\frac{3}{2}}\log\xi + O(B^2\xi^{-\frac{3}{2}}).\end{aligned}\right\} \tag{62}$$

Comparison with (38) and (39) shows that the Pohlhausen method gives results which are in reasonably good agreement with the exact values.

The greatest value of the Pohlhausen method described here lies in its application to the situation when the plate and ambient fluid temperatures are not constant as, for purposes of comparison with the exact solution, we have assumed here. Singh & Cowling give the integral equations corresponding to (41) and (42) when these temperatures are arbitrarily assigned.

REFERENCES

Christian, W. J. 1961 *J. Aero/Space Sci.* **28**, 911.

Cowling, T. G. 1957 *Magnetohydrodynamics.* New York: Interscience.

Geshuni, G. & Zhukovitski, E. M. 1958 *Sov. Phys., J.E.T.P.* **7**, 461.

Gupta, A. S. 1960 *Appl. Sci. Res.* A, **9**, 319.

Gupta, A. S. 1962 *Z.A.M.P.* **13**, 324.

Lykoudis, P. S. 1962 *Int. J. Heat Mass Transfer*, **5**, 23.

Shercliff, J. A. 1953 *Proc. Camb. Phil. Soc.* **49**, 136.

Singh, K. R. & Cowling, T. G. 1963*a* *Q. J. Mech. Appl. Math.* **16**, 1.

Singh, K. R. & Cowling, T. G. 1963*b* *Q. J. Mech. Appl. Math.* **16**, 16.

Stewartson, K. 1957 *J. Math. Phys.* **36**, 173.

Sparrow & Cess 1961 *Int. J. Heat Mass Transfer*, **3**, 267.